

Exact solution of an $su(n)$ spin torus

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Abstract

The trigonometric $su(n)$ spin chain with anti-periodic boundary condition ($su(n)$ spin torus) is demonstrated to be Yang-Baxter integrable. Based on some intrinsic properties of the R -matrix, certain operator product identities of the transfer matrix are derived. These identities and the asymptotic behavior of the transfer matrix together allow us to obtain the exact eigenvalues in terms of an inhomogeneous $T - Q$ relation via the off-diagonal Bethe Ansatz.

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1 Introduction

Study on quantum integrable models [1, 2] has played an essential role in many areas of physics, such as condensed matter physics, quantum field theory, the AdS/CFT [3, 4] correspondence in string theory, nuclear physics, atomic and molecular physics and ultracold atoms. Strikingly, the algebraic Bethe Ansatz (BA) method [5] to solve quantum integrable models with obvious reference states has inspired and led to remarkable developments in different branches of mathematical physics in the past decades. While for quantum integrable models without $U(1)$ -symmetry, obvious reference states are usually absent, making the conventional Bethe Ansatz methods [1, 5, 6, 7, 8, 9] almost inapplicable. Recently, a new approach, i.e., the off-diagonal Bethe Ansatz (ODBA)[10] (for comprehensive introduction we refer the reader to [11]) was proposed to obtain exact solutions of generic integrable models either with or without $U(1)$ symmetry. Several long-standing models were then solved [10, 12, 13, 14, 15, 16, 17, 18] via this method. It should be remarked that some other interesting methods such as the q-Onsager algebra method [19, 20, 21], the modified algebraic Bethe ansatz method [22, 23, 24, 25] and the Sklyanin's separation of variables (SoV) method [26] were also applied to some integrable models related to the $su(2)$ algebra [27, 28, 29, 30].

Quantum spin models provide a typical setting of quantum fluctuations leading to various exotic spin liquid states [31, 32]. The Bethe Ansatz solution [10] of the spin-1/2 chain with anti-periodic boundary conditions ($su(2)$ topological spin torus or quantum Möbius stripe) is a well-known example to reveal topological nature of elementary excitations in such kind of systems. An interesting issue is to study the high-rank systems with topological boundaries. We note that the models of $su(n)$ quantum spin systems are far from merely theoretical exercises: It could be realized either in cold-atom systems in optical lattices [33, 34, 35], in quantum dot arrays [36], or in spin systems with orbital degrees of freedom [37]. The aim of the present work is to study the integrability and exact spectrum of the trigonometric $su(n)$ chain with anti-periodic boundary condition with the nested ODBA [15].

The paper is organized as follows. Section 2 serves as an introduction of our notations and some basic ingredients. The commuting transfer matrix associated with the $su(n)$ spin torus is constructed to show the integrability of the model. In section 3, taking the $su(3)$ spin torus as a concrete example, we derive some operator identities based on intrinsic properties of the R -matrix, which allow us to give the eigenvalues of the transfer matrix in terms of a

nested inhomogeneous $T - Q$ relation. The corresponding Bethe Ansatz equations (BAEs) are also given. The generalization to $su(n)$ case is given in section 4. We summarize our results and give some discussions in Section 5. The generic integrable twisted boundary conditions are shown in Appendix A. Some details about the $su(4)$ case, which could be crucial to understand the procedure for $n \geq 4$, are given in Appendix B.

2 $su(n)$ spin torus

Let \mathbf{V} be an n -dimensional linear space with an orthonormal basis $\{|i\rangle|i = 1, \dots, n\}$, we introduce the Hamiltonian

$$H = \sum_{j=1}^N h_{jj+1}, \quad (2.1)$$

where N is the number of sites and h_{jj+1} is the local Hamiltonian given by

$$h_{jj+1} = \frac{\partial}{\partial u} \{P_{jj+1} R_{jj+1}(u)\}|_{u=0}. \quad (2.2)$$

Here P is the permutation operator on the tensor space $\mathbf{V} \otimes \mathbf{V}$; the R -matrix $R(u) \in \text{End}(\mathbf{V} \otimes \mathbf{V})$ is the trigonometric R -matrix associated with the quantum group [38] $U_q(\widehat{su(n)})$, which was given in [39, 40, 41, 42, 43] and further studied in [44, 45, 46, 47, 48]³

$$\begin{aligned} R(u) = & \sinh(u + \eta) \sum_{k=1}^n E^{k,k} \otimes E^{k,k} + \sinh u \sum_{k \neq l}^n E^{k,k} \otimes E^{l,l} \\ & + \sinh \eta \left(\sum_{k < l}^n e^{\frac{n-2(l-k)}{n}u} + \sum_{k > l}^n e^{-\frac{n-2(k-l)}{n}u} \right) E^{k,l} \otimes E^{l,k}, \end{aligned} \quad (2.3)$$

where the n^2 fundamental matrices $\{E^{k,l}|k, l = 1, \dots, n\}$ are all $n \times n$ matrices with matrix entries $(E^{k,l})_{\beta}^{\alpha} = \delta_{\alpha}^k \delta_{\beta}^l$ and η is the crossing parameter. The R -matrix satisfies the quantum Yang-Baxter equation (QYBE)

$$R_{12}(u_1 - u_2)R_{13}(u_1 - u_3)R_{23}(u_2 - u_3) = R_{23}(u_2 - u_3)R_{13}(u_1 - u_3)R_{12}(u_1 - u_2), \quad (2.4)$$

³The R -matrix given by (2.3) corresponds to the so-called principal gradation, which is related to the R -matrix in homogeneous gradation by some gauge transformation [49].

and possesses the properties:

$$\text{Initial condition :} \quad R_{12}(0) = \sinh \eta P_{12}, \quad (2.5)$$

$$\text{Unitarity :} \quad R_{12}(u)R_{21}(-u) = \rho_1(u) \times \text{id}, \quad \rho_1(u) = -\sinh(u + \eta) \sinh(u - \eta), \quad (2.6)$$

$$\text{Crossing-unitarity :} \quad R_{12}^{t_1}(u)R_{21}^{t_1}(-u - n\eta) = \rho_2(u) \times \text{id}, \quad \rho_2(u) = -\sinh u \sinh(u + n\eta), \quad (2.7)$$

$$\text{Fusion conditions :} \quad R_{12}(-\eta) \propto P_{12}^{(-)}, \quad (2.8)$$

$$\text{Periodicity :} \quad R_{12}(u + i\pi) = -h_1 R_{12}(u) h_1^{-1} = -h_2^{-1} R_{12}(u) h_2. \quad (2.9)$$

Here $R_{21}(u) = P_{12}R_{12}(u)P_{12}$; $P_{12}^{(-)}$ is the q-deformed anti-symmetric project operator in the tensor product space $\mathbf{V} \otimes \mathbf{V}$ (such as below (3.5) and (3.7)); t_i denotes the transposition in the i -th space; h is an $n \times n$ diagonal matrix given by

$$h = \begin{pmatrix} 1 & & & \\ & \omega_n & & \\ & & \ddots & \\ & & & \omega_n^{n-1} \end{pmatrix}, \quad \omega_n = e^{\frac{2i\pi}{n}}, \quad \text{and} \quad h^n = 1. \quad (2.10)$$

Here and below we adopt the standard notation: for any matrix $A \in \text{End}(\mathbf{V})$, A_j is an embedding operator in the tensor space $\mathbf{V} \otimes \mathbf{V} \otimes \cdots$, which acts as A on the j -th space and as an identity on the other factor spaces; $R_{ij}(u)$ is an embedding operator of R -matrix in the tensor space, which acts as an identity on the factor spaces except for the i -th and j -th ones.

In order to construct the quantum spin chain with integrable twisted boundary condition [50], let us introduce an $n \times n$ twist matrix g

$$g = \begin{pmatrix} & & & 1 \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \quad g^n = 1. \quad (2.11)$$

It can be easily checked that the R -matrix (2.3) is invariant with g , namely,

$$R_{12}(u) g_1 g_2 = g_1 g_2 R_{12}(u), \quad (2.12)$$

$$h g = \omega_n g h, \quad (2.13)$$

where h is given by (2.10). (Generic twist matrix satisfying the above equation is given in Appendix A.) This property enables us to construct the integrable $su(n)$ spin torus model.

Similar to the $su(2)$ spin torus (or the XXZ spin chain with anti-periodic boundary condition) [51], the $su(n)$ spin torus is described by the Hamiltonian H given by (2.1) with anti-periodic boundary conditions

$$E_{N+1}^{k,l} = g_1 E_1^{k,l} g_1^{-1}, \quad k, l = 1, \dots, n. \quad (2.14)$$

Let us introduce the “row-to-row” monodromy matrix $T(u)$, an $n \times n$ matrix with operator-valued elements acting on $\mathbf{V}^{\otimes N}$,

$$T_0(u) = R_{0N}(u - \theta_N) R_{0N-1}(u - \theta_{N-1}) \cdots R_{01}(u - \theta_1). \quad (2.15)$$

Here $\{\theta_j | j = 1, \dots, N\}$ are generic free complex parameters which are usually called as inhomogeneity parameters. The transfer matrix $t(u)$ of the associated spin chain describing the Hamiltonian (2.1) with the antiperiodic boundary condition (2.14) can be constructed similarly as [9, 50, 51]

$$t(u) = \text{tr}_0 \{g_0 T_0(u)\}. \quad (2.16)$$

The QYBE (2.4) and the definition (2.15) of the monodromy matrix $T(u)$ imply that the matrix elements of $T(u)$ satisfy the Yang-Baxter algebra:

$$R_{12}(u - v) T_1(u) T_2(v) = T_2(v) T_1(u) R_{12}(u - v). \quad (2.17)$$

The above relation and the invariant relation (2.12) lead to the fact that the transfer matrices $t(u)$ given by (2.16) with different spectral parameters are mutually commuting: $[t(u), t(v)] = 0$. The Hamiltonian (2.1) with the anti-periodic boundary condition (2.14) can be obtained from the transfer matrix as

$$H = \sinh \eta \frac{\partial \ln t(u)}{\partial u} \Big|_{u=0, \{\theta_j\}=0}. \quad (2.18)$$

This ensures the integrability of the $su(n)$ spin torus. The aim of this paper is to obtain eigenvalues of the transfer matrix $t(u)$ specified by the twist matrix (2.11) via ODBA.

3 ODBA solution of the $su(3)$ spin torus

Following the method developed in [15], we apply the fusion techniques [52, 53, 54, 55] to study the $su(n)$ spin torus. For this purpose, besides the fundamental transfer matrix $t(u)$ some other fused transfer matrices $\{t_j(u) | j = 1, \dots, n\}$, which commute with each other and include the original one as $t_1(u) = t(u)$, should be constructed through an anti-symmetric fusion procedure. In this section, we present the results for the $su(3)$ spin torus.

3.1 Operator identities of the transfer matrices

For the $su(3)$ case, the R -matrix $R(u)$ given by (2.3) reads

$$R(u) = \left(\begin{array}{cc|cc|cc} \bar{a}(u) & & & & & \\ & \bar{b}(u) & & & & \\ & & \bar{b}(u) & & & \\ \hline & d(u) & & b(u) & & \\ & & \bar{a}(u) & & & \\ & & & \bar{b}(u) & & \\ \hline & & \bar{c}(u) & & & \\ & & & \bar{d}(u) & & \\ & & & & b(u) & \\ & & & & & \bar{b}(u) \\ & & & & & \bar{a}(u) \end{array} \right) \quad (3.1)$$

with

$$\begin{aligned} \bar{a}(u) &= \sinh(u + \eta), & \bar{b}(u) &= \sinh(u), \\ \bar{c}(u) &= e^{\frac{u}{3}} \sinh(\eta), & \bar{d}(u) &= e^{-\frac{u}{3}} \sinh(\eta), \end{aligned}$$

and the twist matrix g given by (2.11) becomes

$$g = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \text{and } g^3 = 1. \quad (3.2)$$

Following [18], let us introduce the following vectors in the tensor space $\mathbf{V} \otimes \mathbf{V}$ associated with the R -matrix (3.1)

$$\begin{aligned} |\Phi_{12}^{(1)}\rangle &= \frac{1}{\sqrt{2e^{-\frac{\eta}{3}} \cosh(\frac{\eta}{3})}} (|1, 2\rangle - e^{-\frac{\eta}{3}} |2, 1\rangle), \\ |\Phi_{12}^{(2)}\rangle &= \frac{1}{\sqrt{2e^{\frac{\eta}{3}} \cosh(\frac{\eta}{3})}} (|1, 3\rangle - e^{\frac{\eta}{3}} |3, 1\rangle), \\ |\Phi_{12}^{(3)}\rangle &= \frac{1}{\sqrt{2e^{-\frac{\eta}{3}} \cosh(\frac{\eta}{3})}} (|2, 3\rangle - e^{-\frac{\eta}{3}} |3, 2\rangle), \end{aligned} \quad (3.3)$$

and a vector in the tensor space $\mathbf{V} \otimes \mathbf{V} \otimes \mathbf{V}$,

$$\begin{aligned} |\Phi_{123}\rangle &= \frac{1}{\sqrt{6e^{-\frac{\eta}{3}} \cosh(\frac{\eta}{3})}} (|1, 2, 3\rangle - e^{-\frac{\eta}{3}} |1, 3, 2\rangle - e^{-\frac{\eta}{3}} |2, 1, 3\rangle \\ &\quad + |2, 3, 1\rangle + |3, 1, 2\rangle - e^{-\frac{\eta}{3}} |3, 2, 1\rangle). \end{aligned} \quad (3.4)$$

From these vectors we can construct the associated projectors⁴

$$P_{12}^{(-)} = |\Phi_{12}^{(1)}\rangle\langle\Phi_{12}^{(1)}| + |\Phi_{12}^{(2)}\rangle\langle\Phi_{12}^{(2)}| + |\Phi_{12}^{(3)}\rangle\langle\Phi_{12}^{(3)}|, \quad (3.5)$$

$$P_{123}^{(-)} = |\Phi_{123}\rangle\langle\Phi_{123}|. \quad (3.6)$$

Direct calculation shows that the R -matrix given by (3.1) at some degenerate points is proportional to the projectors,

$$R_{12}(-\eta) = P_{12}^{(-)} \times S_{12}^{(-)}, \quad R_{12}(-\eta)R_{13}(-2\eta)R_{23}(-\eta) = P_{123}^{(-)} \times S_{123}^{(-)}, \quad (3.7)$$

where $S_{12}^{(-)} \in \text{End}(\mathbf{V} \otimes \mathbf{V})$ and $S_{123}^{(-)} \in \text{End}(\mathbf{V} \otimes \mathbf{V} \otimes \mathbf{V})$ are some non-degenerate diagonal matrices.

Now we are in position to derive some operator product identities of the transfer matrices which are crucial to obtain eigenvalues of the transfer matrix. Let us evaluate the products of the monodromy matrices at some special points, which lead to the useful relation (for details we refer the readers to the book [11], chapter 7),

$$T_1(\theta_j)T_2(\theta_j - \eta) = P_{21}^{(-)}T_1(\theta_j)T_2(\theta_j - \eta), \quad (3.8)$$

$$T_1(\theta_j)P_{32}^{(-)}T_2(\theta_j - \eta)T_3(\theta_j - 2\eta)P_{32}^{(-)} = P_{321}^{(-)}T_1(\theta_j)T_2(\theta_j - \eta)T_3(\theta_j - 2\eta)P_{32}^{(-)}. \quad (3.9)$$

The invariance (2.12) of the R -matrix $R(u)$ and the relations (3.7) imply the relations

$$[g_1 g_2, P_{21}^{(-)}] = 0 = [g_1 g_2 g_3, P_{321}^{(-)}]. \quad (3.10)$$

With the help of the above relations (3.8)-(3.10), we can calculate the products of the fundamental transfer matrices at some special points

$$\begin{aligned} t(\theta_j)t(\theta_j - \eta) &= \text{tr}_{12}\{g_1 T_1(\theta_j)g_2 T_2(\theta_j - \eta)\} \\ &= \text{tr}_{12}\{g_1 g_2 T_1(\theta_j)T_2(\theta_j - \eta)\} \\ &= \text{tr}_{12}\{g_1 g_2 P_{21}^{(-)}T_1(\theta_j)T_2(\theta_j - \eta)\} \\ &\stackrel{(3.10)}{=} \text{tr}_{12}\{P_{21}^{(-)}g_1 g_2 P_{21}^{(-)}P_{21}^{(-)}T_1(\theta_j)T_2(\theta_j - \eta)P_{21}^{(-)}\} \\ &= \text{tr}_{12}\{g_{<12>}T_{<12>}(\theta_j)\} \\ &= t_2(\theta_j), \end{aligned} \quad (3.11)$$

⁴These operators are q -deformed anti-symmetric projectors and in contrast to the rational ones, $P_{21}^{(-)} = P_{12}P_{12}^{(-)}P_{12} \neq P_{12}^{(-)}$.

where

$$g_{<12>} \equiv P_{21}^{(-)} g_1 g_2 P_{21}^{(-)}, \quad T_{<12>}(u) \equiv P_{21}^{(-)} T_1(u) T_2(u - \eta) P_{21}^{(-)},$$

and the fused transfer matrix $t_2(u)$ is given by

$$t_2(u) = \text{tr}_{12} \{ g_{<12>} T_{<12>}(u) \}. \quad (3.12)$$

Moreover, we can derive

$$\begin{aligned} t(\theta_j) t_2(\theta_j - \eta) &= \text{tr}_{123} \{ g_1 T_1(\theta_j) P_{32}^{(-)} g_2 g_3 P_{32}^{(-)} P_{32}^{(-)} T_2(\theta_j - \eta) T_3(\theta_j - 2\eta) \} \\ &= \text{tr}_{123} \{ g_1 g_{<23>} T_1(\theta_j) P_{32}^{(-)} T_2(\theta_j - \eta) T_3(\theta_j - 2\eta) P_{32}^{(-)} \} \\ &= \text{tr}_{123} \{ g_1 g_2 g_3 T_1(\theta_j) T_{<23>}(\theta_j - \eta) \} \\ &\stackrel{(3.9)}{=} \text{tr}_{123} \{ g_1 g_2 g_3 P_{321}^{(-)} T_1(\theta_j) T_2(\theta_j - \eta) T_3(\theta_j - 2\eta) P_{32}^{(-)} \} \\ &= \text{tr}_{123} \{ g_{<123>} T_{<123>}(\theta_j) \} \\ &= t_3(\theta_j), \end{aligned} \quad (3.13)$$

where

$$\begin{aligned} g_{<123>} &= P_{321}^{(-)} g_1 g_2 g_3 P_{321}^{(-)}, \\ T_{<123>}(u) &= P_{321}^{(-)} T_1(u) T_2(u - \eta) T_3(u - 2\eta) P_{321}^{(-)} \\ &= \prod_{l=1}^N \sinh(u - \theta_l + \eta) \sinh(u - \theta_l - \eta) \sinh(u - \theta_l - 2\eta) P_{321}^{(-)}. \end{aligned}$$

Direct calculation shows that

$$t_3(u) = \text{tr}_{123} \{ g_{<123>} T_{<123>}(u) \} = (-1)^{(3-1)} \text{Det}_q T(u) \times \text{id}, \quad (3.14)$$

where the quantum determinant function $\text{Det}_q T(u)$ reads

$$\text{Det}_q T(u) = \prod_{l=1}^N \sinh(u - \theta_l + \eta) \sinh(u - \theta_l - \eta) \sinh(u - \theta_l - 2\eta).$$

Then the relation (3.13) becomes

$$t(\theta_j) t_2(\theta_j - \eta) = \text{Det}_q T(\theta_j) \times \text{id}. \quad (3.15)$$

Using the initial condition (2.5) and the unitarity relation (2.6), we can derive that the fused transfer matrix $t_2(u)$ vanishes at the points: $\theta_j + \eta$, i.e.,

$$t_2(\theta_j + \eta) = 0, \quad j = 1, \dots, N. \quad (3.16)$$

Moreover, it follows from the fusion procedure and the QYBE (2.4) that the fused transfer matrices constitute commutative families, namely,

$$[t_i(u), t_j(v)] = 0, \quad i, j = 1, 2, 3. \quad (3.17)$$

Now let us consider the periodicities and the asymptotic behaviors of the transfer matrices $t(u)$ and $t_2(u)$. The periodicity (2.9) of the the R -matrix $R(u)$ and the definition (2.15) of the monodromy matrix give rise to the relation

$$T(u + i\pi) = (-1)^N h T(u) h^{-1}, \quad h = \begin{pmatrix} 1 & & \\ & \omega_3 & \\ & & \omega_3^2 \end{pmatrix}, \quad (3.18)$$

with $\omega_3 = e^{\frac{2i\pi}{3}}$. Keeping the fact that $gh = \omega_3 gh$ and using the relation (3.18), we can derive that the transfer matrices $t(u)$ and $t_2(u)$ satisfy the periodicities:

$$t(u + i\pi) = (-1)^N e^{-\frac{2i\pi}{3}} t(u), \quad t_2(u + i\pi) = e^{-\frac{4i\pi}{3}} t_2(u). \quad (3.19)$$

The explicit expression (3.1) of the R -matrix and the definitions (2.15), (2.16) and (3.12) allow us to derive that $e^{-\frac{u}{3}} t(u)$ and $e^{\frac{u}{3}} t_2(u)$, as functions of u , are polynomials of $e^{\pm u}$ with the asymptotic behaviors:

$$e^{-\frac{u}{3}} t(u) \propto e^{\pm(N-1)u} + \dots, \quad u \rightarrow \pm\infty, \quad (3.20)$$

$$e^{\frac{u}{3}} t_2(u) \propto e^{\pm(2N-1)u} + \dots, \quad u \rightarrow \pm\infty. \quad (3.21)$$

3.2 Inhomogeneous $T - Q$ relation and the associated BAEs

The commutativity (3.17) of the transfer matrices $t(u)$ and $t_2(u)$ with different spectral parameters implies that they have common eigenstates. Let $|\Psi\rangle$ be a common eigenstate of $\{t_m(u)\}$, which dose not depend upon u , with the eigenvalues $\Lambda_m(u)$,

$$t_m(u)|\Psi\rangle = \Lambda_m(u)|\Psi\rangle, \quad m = 1, 2, 3.$$

The fusion relations (3.11), (3.15) and (3.16) imply that the eigenvalues $\Lambda_i(u)$ satisfy the relations

$$\Lambda(\theta_j)\Lambda_m(\theta_j - \eta) = \Lambda_{m+1}(\theta_j), \quad m = 1, 2, \quad j = 1, \dots, N, \quad (3.22)$$

$$\Lambda_3(u) = \prod_{l=1}^N \sinh(u - \theta_l + \eta) \prod_{k=1}^2 \sinh(u - \theta_l - k\eta), \quad (3.23)$$

$$\Lambda_2(\theta_j + \eta) = 0, \quad j = 1, \dots, N. \quad (3.24)$$

The periodicity properties (3.19) of the transfer matrices enable us to derive that the eigenvalues $\Lambda_i(u)$ satisfy the associated periodicity relations

$$\Lambda(u + i\pi) = e^{-\frac{2i\pi}{3}}(-1)^N \Lambda(u), \quad \Lambda_2(u + i\pi) = e^{-\frac{4i\pi}{3}} \Lambda_2(u). \quad (3.25)$$

In addition, the asymptotic behaviors (3.20)-(3.21) of the transfer matrices and their definitions (2.16) and (3.12) lead to the fact that the eigenvalues $\Lambda_i(u)$, as a function of u , can be expressed as

$$\Lambda(u) = e^{\frac{u}{3}} \left\{ I_1^{(1)} e^{(N-1)u} + I_2^{(1)} e^{(N-3)u} + \dots + I_N^{(1)} e^{-(N-1)u} \right\}, \quad (3.26)$$

$$\Lambda_2(u) = e^{-\frac{u}{3}} \left\{ I_1^{(2)} e^{(2N-1)u} + I_2^{(2)} e^{(2N-3)u} + \dots + I_{2N}^{(2)} e^{-(2N-1)u} \right\}, \quad (3.27)$$

where $\{I_j^{(1)} | j = 1, \dots, N\}$ and $\{I_j^{(2)} | j = 1, \dots, 2N\}$ are $3N$ constants which are eigenstate dependent. Then these constants can be completely determined by the $3N$ equations (3.22)-(3.24). The above relations (3.22)-(3.27) allow us to express the eigenvalues $\Lambda_i(u)$ in terms of some inhomogeneous $T - Q$ relations [11].

For this purpose, let us introduce some functions:

$$a(u) = \prod_{l=1}^N \sinh(u - \theta_l + \eta), \quad d(u) = \prod_{l=1}^N \sinh(u - \theta_l) = a(u - \eta), \quad (3.28)$$

$$Q^{(i)}(u) = \prod_{l=1}^{N_i} \sinh(u - \lambda_l^{(i)}), \quad i = 1, 2, 3, 4,$$

$$f_1(u) = f_1^{(+)} e^u + f_1^{(-)} e^{-u}, \quad f_2(u) = f_2^{(-)} e^{-u},$$

where the $(N_1 + N_2 + N_3 + N_4)$ parameters $\{\lambda_l^{(i)} | l = 1, \dots, N_i; i = 1, 2, 3, 4\}$ and the parameters $f_1^{(\pm)}$ and $f_2^{(-)}$ will be specified later by the associated BAEs (3.35)-(3.42). For

convenience, we introduce further the following notations:

$$\begin{aligned}
Z_1(u) &= e^{\phi_1} e^{\frac{4u}{3}} a(u) \frac{Q^{(1)}(u - \eta)}{Q^{(2)}(u)}, \\
Z_2(u) &= e^{-\phi_1} \omega_3 e^{-\frac{2(u+\eta)}{3}} d(u) \frac{Q^{(2)}(u + \eta) Q^{(3)}(u - \eta)}{Q^{(1)}(u) Q^{(4)}(u)}, \\
Z_3(u) &= \omega_3^2 e^{-\frac{2(u+2\eta)}{3}} d(u) \frac{Q^{(4)}(u + \eta)}{Q^{(3)}(u)}, \\
X_1(u) &= e^{\frac{u}{3}} a(u) d(u) \frac{Q^{(3)}(u - \eta) f_1(u)}{Q^{(1)}(u) Q^{(2)}(u)}, \\
X_2(u) &= e^{\frac{u}{3}} a(u) d(u) \frac{Q^{(2)}(u + \eta) f_2(u)}{Q^{(3)}(u) Q^{(4)}(u)}, \tag{3.29}
\end{aligned}$$

where ϕ_1 is a parameter to be determined later. The eigenvalues $\{\Lambda_i(u)\}$ satisfying (3.22)-(3.24) and (3.26)-(3.27) can be expressed in terms of the inhomogeneous $T - Q$ relations as

follows ⁵

$$\begin{aligned}
\Lambda(u) &= Z_1(u) + Z_2(u) + Z_3(u) + X_1(u) + X_2(u) \\
&= e^{\frac{u}{3}} \left\{ e^{\phi_1} e^u a(u) \frac{Q^{(1)}(u-\eta)}{Q^{(2)}(u)} + e^{-\phi_1} \omega_3 e^{-u-\frac{2\eta}{3}} d(u) \frac{Q^{(2)}(u+\eta) Q^{(3)}(u-\eta)}{Q^{(1)}(u) Q^{(4)}(u)} \right. \\
&\quad + \omega_3^2 e^{-u-\frac{4\eta}{3}} d(u) \frac{Q^{(4)}(u+\eta)}{Q^{(3)}(u)} + a(u) d(u) \frac{Q^{(3)}(u-\eta) f_1(u)}{Q^{(1)}(u) Q^{(2)}(u)} \\
&\quad \left. + a(u) d(u) \frac{Q^{(2)}(u+\eta) f_2(u)}{Q^{(3)}(u) Q^{(4)}(u)} \right\}, \tag{3.30}
\end{aligned}$$

$$\begin{aligned}
\Lambda_2(u) &= Z_1(u) Z_2^{(1)}(u) + Z_1(u) Z_3^{(1)}(u) + Z_2(u) Z_3^{(1)}(u) + X_1(u) Z_3^{(1)}(u) + Z_1(u) X_2^{(1)}(u) \\
&= e^{-\frac{u}{3}} d(u-\eta) \left\{ \omega_3 e^u a(u) \frac{Q^{(3)}(u-2\eta)}{Q^{(4)}(u-\eta)} + e^{-\phi_1} e^{-\frac{4\eta}{3}} e^{-u} d(u) \frac{Q^{(2)}(u+\eta)}{Q^{(1)}(u)} \right. \\
&\quad + \omega_3^2 e^{\phi_1} e^{-\frac{2\eta}{3}} e^u a(u) \frac{Q^{(1)}(u-\eta) Q^{(4)}(u)}{Q^{(2)}(u) Q^{(3)}(u-\eta)} + \omega_3^2 e^{-\frac{2\eta}{3}} a(u) d(u) \frac{Q^{(4)}(u) f_1(u)}{Q^{(1)}(u) Q^{(2)}(u)} \\
&\quad \left. + e^{\phi_1} e^{2u-\frac{\eta}{3}} a(u) d(u) \frac{Q^{(1)}(u-\eta) f_2(u-\eta)}{Q^{(3)}(u-\eta) Q^{(4)}(u-\eta)} \right\}, \tag{3.31}
\end{aligned}$$

$$\begin{aligned}
\Lambda_3(u) &= Z_1(u) Z_2^{(1)}(u) Z_3^{(2)}(u) \\
&= \omega_3^3 a(u) d(u-\eta) d(u-2\eta) = a(u) d(u-\eta) d(u-2\eta). \tag{3.32}
\end{aligned}$$

Here and below we adopt the conventions

$$Z_i^{(l)}(u) = Z_i(u - l\eta), \quad X_i^{(l)}(u) = X_i(u - l\eta). \tag{3.33}$$

To make the inhomogeneous $T - Q$ relations (3.30) and (3.31) to fulfill the asymptotic behavior (3.26)-(3.27), the non-negative integers N_i should satisfy the relations

$$N_1 = N_2 = N_3 = N_4 = N, \tag{3.34}$$

and the $4N + 4$ parameters $\{\lambda_l^{(i)} | l = 1, \dots, N; i = 1, 2, 3, 4\}$, $f_1^{(\pm)}$, $f_2^{(-)}$ and e^{ϕ_1} satisfy the

⁵It is well-known that for the $su(n)$ spin chain with the periodic boundary condition the eigenvalues are given by the usual homogeneous $T - Q$ relations which only have $n - 1$ types of Q -functions. However, it seems that for the anti-periodic boundary condition case the associated inhomogeneous $T - Q$ relations (3.30)-(3.31) (or (4.12) for the generic n) have to involve more types of Q -functions (c.f. $2(n - 1)$ types of Q -functions see (4.12) below) than that of the periodic case, except the $n = 2$ case [10].

associated BAEs⁶:

$$\omega_3 e^{-\phi_1} e^{-\lambda_j^{(1)} - \frac{2\eta}{3}} \frac{Q^{(2)}(\lambda_j^{(1)} + \eta)}{Q^{(4)}(\lambda_j^{(1)})} + a(\lambda_j^{(1)}) \frac{f_1(\lambda_j^{(1)})}{Q^{(2)}(\lambda_j^{(1)})} = 0, \quad j = 1, \dots, N, \quad (3.35)$$

$$e^{\phi_1} e^{\lambda_j^{(2)}} Q^{(1)}(\lambda_j^{(2)} - \eta) + d(\lambda_j^{(2)}) \frac{Q^{(3)}(\lambda_j^{(2)} - \eta) f_1(\lambda_j^{(2)})}{Q^{(1)}(\lambda_j^{(2)})} = 0, \quad j = 1, \dots, N, \quad (3.36)$$

$$\omega_3^2 e^{-\lambda_j^{(3)} - \frac{4\eta}{3}} Q^{(4)}(\lambda_j^{(3)} + \eta) + a(\lambda_j^{(3)}) \frac{Q^{(2)}(\lambda_j^{(3)} + \eta) f_2(\lambda_j^{(3)})}{Q^{(4)}(\lambda_j^{(3)})} = 0, \quad j = 1, \dots, N, \quad (3.37)$$

$$\omega_3 e^{-\phi_1} e^{-\lambda_j^{(4)} - \frac{2\eta}{3}} \frac{Q^{(3)}(\lambda_j^{(4)} - \eta)}{Q^{(1)}(\lambda_j^{(4)})} + a(\lambda_j^{(4)}) \frac{f_2(\lambda_j^{(4)})}{Q^{(3)}(\lambda_j^{(4)})} = 0, \quad j = 1, \dots, N, \quad (3.38)$$

$$e^{\phi_1} e^{-\Theta - \chi^{(1)} + \chi^{(2)}} + e^{-2\Theta + \chi^{(1)} + \chi^{(2)} - \chi^{(3)}} f_1^{(+)} = 0, \quad (3.39)$$

$$\omega_3 e^{-\phi_1} e^{-\frac{2\eta}{3} + \Theta - \chi^{(1)} + \chi^{(2)} + \chi^{(3)} - \chi^{(4)}} + \omega_3^2 e^{-\frac{4\eta}{3} + \Theta - \chi^{(3)} + \chi^{(4)} - N\eta} \\ + e^{2\Theta - N\eta} \left\{ e^{-\chi^{(1)} - \chi^{(2)} + \chi^{(3)} + N\eta} f_1^{(-)} + e^{+\chi^{(2)} - \chi^{(3)} - \chi^{(4)} - N\eta} f_2^{(-)} \right\} = 0, \quad (3.40)$$

$$\omega_3 e^{-\Theta - \chi^{(3)} + \chi^{(4)}} + \omega_3^2 e^{\phi_1} e^{-\frac{2\eta}{3} - \Theta - \chi^{(1)} + \chi^{(2)} + \chi^{(3)} - \chi^{(4)} + N\eta} \\ + e^{-2\Theta + N\eta} \left\{ \omega_3^2 e^{-\frac{2\eta}{3} + \chi^{(1)} + \chi^{(2)} - \chi^{(4)}} f_1^{(+)} + e^{\phi_1} e^{\frac{2\eta}{3} - \chi^{(1)} + \chi^{(3)} + \chi^{(4)} + N\eta} f_2^{(-)} \right\} = 0, \quad (3.41)$$

$$e^{-\phi_1} e^{-\frac{4\eta}{3} + \Theta - \chi^{(1)} + \chi^{(2)} - N\eta} + \omega_3^2 e^{-\frac{2\eta}{3} + 2\Theta - \chi^{(1)} - \chi^{(2)} + \chi^{(4)} - N\eta} f_1^{(-)} = 0, \quad (3.42)$$

where

$$\Theta = \sum_{l=1}^N \theta_l, \quad \chi^{(i)} = \sum_{l=1}^N \lambda_l^{(i)}, \quad i = 1, 2, 3, 4.$$

It is easy to check that the inhomogeneous $T - Q$ relations (3.30) and (3.31) fulfill the relations (3.22)-(3.24) and the periodicity properties (3.25). The BEAs (3.39)-(3.42) ensure that the inhomogeneous $T - Q$ relations (3.30) and (3.31) indeed satisfy the asymptotic behavior (3.26)-(3.27), while the BAEs (3.35)-(3.38) assure that the inhomogeneous $T - Q$ relations have no singularity at points $\lambda_l^{(i)}$.

⁶It is still an interesting open problem to investigate the structure of the Bethe roots of the BAEs (3.35)-(3.42) for a large N . One promising strategy might be to study the corresponding elliptical model for the large sites with the crossing parameter taken some special values (which will become dense in the whole complex plan when $N \rightarrow \infty$) for which the inhomogeneous $T - Q$ relation reduce to the usual one [13]. This allows one to study the pattern of the corresponding Bethe roots for the large N and then taking the trigonometric limit we can obtain the pattern of the Bethe root for the trigonometric models for a large N . This strategy has proven to be very successful for the studying the thermodynamics of the spin- $\frac{1}{2}$ open XXZ chain [16].

4 Results for the $su(n)$ spin torus

For the $su(n)$ case, by using the similar method introduced in the previous section, we can derive that the fused transfer matrices $\{t_j(u)|j = 1, \dots, n\}$ satisfy the analogous operator product identities such as (3.11), (3.15), (3.16), (3.19) and (3.20)-(3.21). These identities lead to that the corresponding eigenvalues $\{\Lambda_j(u)|j = 1, \dots, n\}$ satisfy the functional relations:

$$\Lambda(\theta_j)\Lambda_m(\theta_j - \eta) = \Lambda_{m+1}(\theta_j), \quad m = 1, \dots, n-1, \quad j = 1, \dots, N, \quad (4.1)$$

$$\Lambda_m(\theta_j + k\eta) = 0, \quad k = 1, \dots, m-1, \quad m = 1, \dots, n-1, \quad j = 1, \dots, N, \quad (4.2)$$

$$\Lambda_n(u) = (-1)^{n-1} \prod_{l=1}^N \sinh(u - \theta_l + \eta) \prod_{k=1}^{n-1} \sinh(u - \theta_l - k\eta) \times \text{id}, \quad (4.3)$$

$$\Lambda_m(u + i\pi) = e^{-m(\frac{2}{n})i\pi} ((-1)^N)^m \Lambda_m(u), \quad m = 1, \dots, n-1, \quad (4.4)$$

$$e^{-u+2(\frac{m}{n})u} \Lambda_m(u) \propto e^{\pm(mN-1)u} + \dots, \quad u \rightarrow \pm\infty, \quad m = 1, \dots, n-1. \quad (4.5)$$

Similar to the $su(3)$ case, the above relations completely determine the eigenvalues $\Lambda_i(u)$ and thus enable us to express them in terms of certain inhomogeneous $T - Q$ relations as those given by (3.30)-(3.31). For the $su(n)$ case, let us introduce the functions:

$$Q^{(i)}(u) = \prod_{l=1}^{N_i} \sinh(u - \lambda_l^{(i)}), \quad i = 1, \dots, 2n-2, \quad (4.6)$$

$$Z_1(u) = e^{\phi_1} e^{(2-\frac{2}{n})u} a(u) \frac{Q^{(1)}(u - \eta)}{Q^{(2)}(u)},$$

$$Z_2(u) = e^{\phi_2} \omega_n e^{-\frac{2(u+\eta)}{n}} d(u) \frac{Q^{(2)}(u + \eta) Q^{(3)}(u - \eta)}{Q^{(1)}(u) Q^{(4)}(u)},$$

\vdots

$$Z_i(u) = e^{\phi_i} \omega_n^{i-1} e^{-\frac{2(u+(i-1)\eta)}{n}} d(u) \frac{Q^{(2i-2)}(u + \eta) Q^{(2i-1)}(u - \eta)}{Q^{(2i-3)}(u) Q^{(2i)}(u)},$$

\vdots

$$Z_{n-1}(u) = e^{-\sum_{j=1}^{n-2} \phi_j} \omega_n^{n-2} e^{-\frac{2(u+(n-2)\eta)}{n}} d(u) \frac{Q^{(2n-4)}(u + \eta) Q^{(2n-3)}(u - \eta)}{Q^{(2n-5)}(u) Q^{(2n-2)}(u)},$$

$$Z_n(u) = \omega_n^{n-1} e^{-\frac{2(u+(n-1)\eta)}{n}} d(u) \frac{Q^{(2n-2)}(u + \eta)}{Q^{(2n-3)}(u)},$$

and

$$\begin{aligned}
X_1(u) &= e^{(1-\frac{2}{n})u} a(u) d(u) \frac{Q^{(3)}(u-\eta) f_1(u)}{Q^{(1)}(u) Q^{(2)}(u)}, \\
X_2(u) &= e^{(1-\frac{2}{n})u} a(u) d(u) \frac{Q^{(2)}(u+\eta) Q^{(5)}(u-\eta) f_2(u)}{Q^{(3)}(u) Q^{(4)}(u)}, \\
&\vdots \\
X_i(u) &= e^{(1-\frac{2}{n})u} a(u) d(u) \frac{Q^{(2i-2)}(u+\eta) Q^{(2i+1)}(u-\eta) f_i(u)}{Q^{(2i-1)}(u) Q^{(2i)}(u)}, \\
&\vdots \\
X_{n-1}(u) &= e^{(1-\frac{2}{n})u} a(u) d(u) \frac{Q^{(2n-4)}(u+\eta) f_{n-1}(u)}{Q^{(2n-3)}(u) Q^{(2n-2)}(u)}. \tag{4.7}
\end{aligned}$$

Here $\omega_n = e^{\frac{2i\pi}{n}}$ such that $\omega_n^n = 1$, the functions $\{f_i(u) | i = 1, \dots, n-1\}$ are

$$f_1(u) = f_1^{(+)} e^u + f_1^{(-)} e^{-u}, \tag{4.8}$$

$$f_i(u) = f_i^{(-)} e^{-u}, \quad i = 2, \dots, n-1. \tag{4.9}$$

The $2(n-1)$ constants $f_1^{(\pm)}$, $\{f_i^{(-)} | i = 2, \dots, n-1\}$ and $\{\phi_i, i = 1, \dots, n-2\}$ are to be determined later. We define further functions $\{Y_l(u) | l = 1, \dots, 2n-1\}$,

$$\begin{cases} Y_{2j-1}(u) = Z_j(u), & j = 1, \dots, n, \\ Y_{2j}(u) = X_j(u), & j = 1, \dots, n-1, \end{cases} \tag{4.10}$$

and take the notation

$$Y_j^{(l)}(u) = Y_j(u - l\eta), \quad l = 1, \dots, n, \quad j = 1, \dots, 2n-1. \tag{4.11}$$

The eigenvalue $\{\Lambda_m(u) | m = 1, \dots, n-1\}$ satisfying the relations (4.1)-(4.5) can be given in terms of the inhomogeneous $T-Q$ relations as ⁷

$$\Lambda_m(u) = \sum'_{1 \leq i_1 < i_2 < \dots < i_m \leq 2n-1} Y_{i_1}(u) Y_{i_2}^{(1)}(u) \dots Y_{i_m}^{(m-1)}(u), \quad m = 1, \dots, n-1. \tag{4.12}$$

The sum \sum' is over the constrained increasing sequences $1 \leq i_1 < i_2 < \dots < i_m \leq 2n-1$ such that when any $i_k = 2j$ (i.e., $Y_{i_k}^{(k-1)}(u) = Y_{2j}^{(k-1)}(u) = X_j^{(k-1)}(u)$), then $i_{k-1} \leq 2j-3$ and

⁷For the $n=2$ case, the corresponding inhomogeneous $T-Q$ relation reduces to the alternative inhomogeneous $T-Q$ given in [11] (i.e., the equation (4.4.1) of subchapter 4.4).

$i_{k+1} \geq 2j + 3$. Namely, when $Y_{i_k}(u) = X_j(u)$, the previous element $Y_{i_{k-1}}(u)$ and the next element $Y_{i_{k+1}}(u)$ can not be chosen as its nearest neighbors (e.g., $X_{j-1}(u)$, $Z_j(u)$, $Z_{j+1}(u)$ and $X_{j+1}(u)$) in the diagram (4.13).

$$\begin{array}{ccccccc}
 Z_1 & \text{---} & Z_2 & \text{---} & Z_3 & \text{---} & Z_4 \cdots \cdots \cdots Z_n \\
 & \diagdown & / & \diagdown & / & \diagdown & / \\
 & X_1 & \text{---} & X_2 & \text{---} & X_3 \cdots \cdots \cdots X_{n-1}
 \end{array} \quad (4.13)$$

To satisfy the asymptotic behaviors (4.5), the non-negative integers $\{N_i | i = 1, \dots, 2(n-1)\}$ must be chosen as follows:

- For odd n ,

$$N_{2i-1} = N_{2i} = N_{2(n-i)-1} = N_{2(n-i)} = \frac{i(n-i)}{2}N, \quad i = 1, \dots, \frac{n-1}{2}. \quad (4.14)$$

- For even n and even N ,

$$N_{2i-1} = N_{2i} = N_{2(n-i)-1} = N_{2(n-i)} = \frac{i(n-i)}{2}N, \quad i = 1, \dots, \frac{n}{2}. \quad (4.15)$$

- For even n and odd N ,

$$N_{2i-1} = N_{2i} = N_{2(n-i)-1} = N_{2(n-i)} = \frac{i(n-i)}{2}N + \frac{i}{2}, \quad i = 1, \dots, \frac{n}{2}. \quad (4.16)$$

In contrast with (4.9), the function $f_{\frac{n}{2}}(u)$ now should be adjusted to

$$f_{\frac{n}{2}}(u) = \sinh(u) f_{\frac{n}{2}}^{(-)} e^{-u}. \quad (4.17)$$

We present the details for the $su(4)$ case in Appendix B, which could be crucial to understand the structure for $n \geq 4$.

It is easy to check that if $\{N_i | i = 1, \dots, 2(n-1)\}$ are chosen as (4.14)-(4.16) the corresponding inhomogeneous $T - Q$ relations (4.12) have the asymptotic behavior

$$\begin{aligned}
 e^{-u+2(\frac{m}{n})u} \Lambda_m(u) &= F_m^{(\pm)} e^{\pm(mN+1)u} + F_m^{(\pm)'} e^{\pm(mN-1)u} + \dots, \quad u \rightarrow \pm\infty, \\
 m &= 1, \dots, n-1.
 \end{aligned} \quad (4.18)$$

The $2(n-1)$ coefficients $F_m^{(\pm)}$ are the functions of the parameters $\{\lambda_j^{(i)} | i = 1, \dots, 2(n-1); j = 1, \dots, N_i\}$, $f_1^{(\pm)}$, $\{f_i^{(-)} | i = 2, \dots, n-1\}$ and $\{\phi_i, i = 1, \dots, n-2\}$. The explicit expressions

can be obtained by direct calculation. To make (4.5) satisfied, the coefficients $F_m^{(\pm)}$ must vanish which leads to the $2(n-1)$ BAEs (for an example, (3.39)-(3.42) for the $su(3)$ case)

$$F_m^{(\pm)} = 0, \quad m = 1, \dots, n-1. \quad (4.19)$$

Moreover, the vanishing condition of the residues of $\Lambda_m(u)$ at the points $\lambda_j^{(i)}$ gives rise to the other BAEs:

$$e^{\phi_2} \omega_n e^{-\lambda_j^{(1)} - \frac{2\eta}{n}} \frac{Q^{(2)}(\lambda_j^{(1)} + \eta)}{Q^{(4)}(\lambda_j^{(1)})} + a(\lambda_j^{(1)}) \frac{f_1(\lambda_j^{(1)})}{Q^{(2)}(\lambda_j^{(1)})} = 0, \quad j = 1, \dots, N_1, \quad (4.20)$$

$$e^{\phi_1} e^{\lambda_j^{(2)}} Q^{(1)}(\lambda_j^{(2)} - \eta) + d(\lambda_j^{(2)}) \frac{Q^{(3)}(\lambda_j^{(2)} - \eta) f_1(\lambda_j^{(2)})}{Q^{(1)}(\lambda_j^{(2)})} = 0, \quad j = 1, \dots, N_2, \quad (4.21)$$

$$e^{\phi_i} \omega_n^{i-1} e^{-\lambda_j^{(2i)} - \frac{2(i-1)\eta}{n}} \frac{Q^{(2i-1)}(\lambda_j^{(2i)} - \eta)}{Q^{(2i-3)}(\lambda_j^{(2i)})} + a(\lambda_j^{(2i)}) \frac{Q^{(2i+1)}(\lambda_j^{(2i)} - \eta) f_i(\lambda_j^{(2i)})}{Q^{(2i-1)}(\lambda_j^{(2i)})} = 0, \\ i = 2, \dots, n-2, \quad j = 1, \dots, N_{2i}, \quad (4.22)$$

$$e^{\phi_{i+1}} \omega_n^i e^{-\lambda_j^{(2i+1)} - \frac{2i\eta}{n}} \frac{Q^{(2i)}(\lambda_j^{(2i+1)} + \eta)}{Q^{(2i+2)}(\lambda_j^{(2i+1)})} + a(\lambda_j^{(2i+1)}) \frac{Q^{(2i-2)}(\lambda_j^{(2i+1)} + \eta) f_i(\lambda_j^{(2i+1)})}{Q^{(2i)}(\lambda_j^{(2i+1)})} = 0, \\ i = 2, \dots, n-3, \quad j = 1, \dots, N_{2i-1}, \quad (4.23)$$

$$e^{-\sum_{j=1}^{n-2} \phi_j} \omega_n^{n-2} e^{-\lambda_j^{(2n-5)} - \frac{2(n-2)\eta}{n}} \frac{Q^{(2n-4)}(\lambda_j^{(2n-5)} + \eta)}{Q^{(2n-2)}(\lambda_j^{(2n-5)})} \\ + a(\lambda_j^{(2n-5)}) \frac{Q^{(2n-6)}(\lambda_j^{(2n-5)} + \eta) f_{n-2}(\lambda_j^{(2n-5)})}{Q^{(2n-4)}(\lambda_j^{(2n-5)})} = 0, \quad j = 1, \dots, N_{2n-5}, \quad (4.24)$$

$$\omega_n^{n-1} e^{-\lambda_j^{(2n-3)} - \frac{2(n-1)\eta}{n}} Q^{(2n-2)}(\lambda_j^{(2n-3)} + \eta) + a(\lambda_j^{(2n-3)}) \frac{Q^{(2n-4)}(\lambda_j^{(2n-3)} + \eta) f_{n-1}(\lambda_j^{(2n-3)})}{Q^{(2n-2)}(\lambda_j^{(2n-3)})} = 0, \\ j = 1, \dots, N_{2n-3}, \quad (4.25)$$

$$e^{-\sum_{j=1}^{n-2} \phi_j} \omega_n^{n-2} e^{-\lambda_j^{(2n-2)} - \frac{2(n-2)\eta}{n}} \frac{Q^{(2n-3)}(\lambda_j^{(2n-2)} - \eta)}{Q^{(2n-5)}(\lambda_j^{(2n-2)})} + a(\lambda_j^{(2n-2)}) \frac{f_{n-1}(\lambda_j^{(2n-2)})}{Q^{(2n-3)}(\lambda_j^{(2n-2)})} = 0, \\ j = 1, \dots, N_{2n-2}. \quad (4.26)$$

Associated with the BAEs (4.19)-(4.26), the inhomogeneous $T - Q$ relation (4.12) give the eigenvalues of the transfer matrix of the $su(n)$ spin torus.

5 Conclusions

In this paper, we have studied the $su(n)$ spin torus described by the Hamiltonian (2.1) with the anti-periodic boundary condition (2.14). In the framework of ODBA, we have obtained the eigenvalues of the corresponding transfer matrix in terms of the inhomogeneous $T - Q$ relation (4.12) and the associated BAEs (4.19)-(4.26). The exact spectrum obtained in this paper allows us further to construct the corresponding eigenstates. The results will be presented elsewhere [56].

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Appendix A: Generic twist matrices

A generic twist matrix \mathcal{G} associated with an integrable boundary satisfies the relation

$$R_{12}(u-v) \mathcal{G}_1 \mathcal{G}_2 = \mathcal{G}_1 \mathcal{G}_2 R_{12}(u-v). \quad (\text{A.1})$$

Normally, \mathcal{G} is a c-number matrix. The solutions of (A.1) with the R -matrix given by (2.3) can be specified as n classes labeled by l

$$\mathcal{D} g^l, \quad l = 0, \dots, n-1, \quad (\text{A.2})$$

where \mathcal{D} is an arbitrary non-degenerate diagonal $n \times n$ matrix and g is given by (2.11). All solutions to (A.1) are some products of elements of these classes. $\mathcal{G} = \mathcal{D}$ corresponds to the diagonal twisted boundary condition (including the periodic boundary condition $\mathcal{D} = \text{id}$ as a special case). Without losing the generality, in this paper we consider the twist matrix $\mathcal{G} = g$ which corresponds to the antiperiodic boundary condition (2.14). The generalization to the other cases is straightforward.

Appendix B: $T - Q$ relation for the $su(4)$ spin torus

For $n = 4$, the functions (4.6)-(4.7) and the functions $X_j(u)$ read

$$Q^{(i)}(u) = \prod_{l=1}^{N_i} \sinh(u - \lambda_l^{(i)}), \quad i = 1, \dots, 6, \quad (\text{B.1})$$

$$Z_1(u) = e^{\phi_1} e^{\frac{3u}{2}} a(u) \frac{Q^{(1)}(u - \eta)}{Q^{(2)}(u)},$$

$$Z_2(u) = e^{\phi_2} \omega_4 e^{-\frac{u+\eta}{2}} d(u) \frac{Q^{(2)}(u + \eta) Q^{(3)}(u - \eta)}{Q^{(1)}(u) Q^{(4)}(u)},$$

$$Z_3(u) = e^{-\phi_1 - \phi_2} \omega_4^2 e^{-\frac{u}{2} - \eta} d(u) \frac{Q^{(4)}(u + \eta) Q^{(5)}(u - \eta)}{Q^{(3)}(u) Q^{(6)}(u)},$$

$$Z_4(u) = \omega_4^3 e^{-\frac{u+3\eta}{2}} d(u) \frac{Q^{(6)}(u + \eta)}{Q^{(5)}(u)},$$

$$X_1(u) = e^{\frac{u}{2}} a(u) d(u) \frac{Q^{(3)}(u - \eta) f_1(u)}{Q^{(1)}(u) Q^{(2)}(u)},$$

$$X_2(u) = e^{\frac{u}{2}} a(u) d(u) \frac{Q^{(2)}(u + \eta) Q^{(5)}(u - \eta) f_2(u)}{Q^{(3)}(u) Q^{(4)}(u)},$$

$$X_3(u) = e^{\frac{u}{2}} a(u) d(u) \frac{Q^{(4)}(u + \eta) f_3(u)}{Q^{(5)}(u) Q^{(6)}(u)}. \quad (\text{B.2})$$

Here $\omega_4 = e^{\frac{2i\pi}{4}}$. The corresponding $T - Q$ relations (4.12) become

$$\Lambda(u) = Z_1(u) + Z_2(u) + Z_3(u) + Z_4(u) + X_1(u) + X_2(u) + X_3(u), \quad (\text{B.3})$$

$$\begin{aligned} \Lambda_2(u) = & Z_1(u)Z_2(u - \eta) + Z_1(u)X_2(u - \eta) + Z_1(u)Z_3(u - \eta) + Z_1(u)X_3(u - \eta) \\ & + Z_1(u)Z_4(u - \eta) + X_1(u)Z_3(u - \eta) + X_1(u)X_3(u - \eta) + X_1(u)Z_4(u - \eta) \\ & + Z_2(u)Z_3(u - \eta) + Z_2(u)X_3(u - \eta) + Z_2(u)Z_4(u - \eta) + X_2(u)Z_4(u - \eta) \\ & + Z_3(u)Z_4(u - \eta), \end{aligned} \quad (\text{B.4})$$

$$\begin{aligned} \Lambda_3(u) = & Z_1(u)Z_2(u - \eta)Z_3(u - 2\eta) + Z_1(u)Z_2(u - \eta)X_3(u - 2\eta) \\ & + Z_1(u)Z_2(u - \eta)Z_4(u - 2\eta) + Z_1(u)X_2(u - \eta)Z_4(u - 2\eta) \\ & + Z_1(u)Z_3(u - \eta)Z_4(u - 2\eta) + X_1(u)Z_3(u - \eta)Z_4(u - 2\eta) \\ & + Z_2(u)Z_3(u - \eta)Z_4(u - 2\eta), \end{aligned} \quad (\text{B.5})$$

$$\Lambda_4(u) = Z_1(u)Z_2(u - \eta)Z_3(u - 2\eta)Z_4(u - 3\eta). \quad (\text{B.6})$$

- For even N ,

$$N_1 = N_2 = N_5 = N_6 = \frac{3}{2}N, \quad N_3 = N_4 = 2N, \quad (\text{B.7})$$

and the functions $f_i(u)$ are :

$$f_1(u) = f_1^{(+)}e^u + f_1^{(-)}e^{-u}, \quad f_2(u) = f_2^{(-)}e^{-u}, \quad f_3(u) = f_3^{(-)}e^{-u}. \quad (\text{B.8})$$

- For odd N ,

$$N_1 = N_2 = N_5 = N_6 = \frac{3N+1}{2}, \quad N_3 = N_4 = 2N+1, \quad (\text{B.9})$$

and the functions $f_i(u)$ are :

$$f_1(u) = f_1^{(+)}e^u + f_1^{(-)}e^{-u}, \quad f_2(u) = \sinh(u) f_2^{(-)}e^{-u}, \quad f_3(u) = f_3^{(-)}e^{-u}. \quad (\text{B.10})$$

References

- [1] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic Press, 1982.

- [2] V. E. Korepin, N. M. Bogoliubov and A. G. Izergin, *Quantum Inverse Scattering Method and Correlation Function*, Cambridge University Press, 1993.
- [3] J. M. Maldacena, *Adv. Theor. Math. Phys.* **2** (1998), 231.
- [4] N. Beisert, C. Ahn, L. F. Alday, Z. Bajnok, J. M. Drummond, et al., *Lett. Math. Phys.* **99** (2012), 1.
- [5] L. A. Takhtadzhan and L. D. Faddeev, *Russ. Math. Surveys* **34** (1979), 11.
- [6] H. Bethe, *Z. Phys.* **71** (1931), 205.
- [7] E. K. Sklyanin and L. D. Faddeev, *Sov. Phys. Dokl.* **23** (1978), 902.
- [8] F. C. Alcaraz, M. N. Barber, M. T. Batchelor, R. J. Baxter and G. R. W. Quispel, *J. Phys. A* **20** (1987), 6397.
- [9] E. K. Sklyanin, *J. Phys. A* **21** (1988), 2375.
- [10] J. Cao, W.-L. Yang, K. Shi and Y. Wang, *Phys. Rev. Lett.* **111** (2013), 137201.
- [11] Y. Wang, W.-L. Yang, J. Cao and K. Shi, *Off-Diagonal Bethe Ansatz for Exactly Solvable Models*, Springer Press, 2015.
- [12] J. Cao, W.-L. Yang, K. Shi and Y. Wang, *Nucl. Phys. B* **875** (2013), 152.
- [13] J. Cao, S. Cui, W.-L. Yang, K. Shi and Y. Wang, *Nucl. Phys. B* **886** (2014), 185.
- [14] J. Cao, W.-L. Yang, K. Shi and Y. Wang, *Nucl. Phys. B* **877** (2013), 152.
- [15] J. Cao, W.-L. Yang, K. Shi and Y. Wang, *JHEP* **04** (2014), 143.
- [16] Y.-Y. Li, J. Cao, W.-L. Yang, K. Shi, Y. Wang, *Nucl. Phys. B* **879** (2014), 98.
- [17] X. Zhang, J. Cao, W.-L. Yang, K. Shi, Y. Wang, *J. Stat. Mech.* (2014), P04031.
- [18] K. Hao, J. Cao, G.-L. Li, W.-L. Yang, K. Shi and Y. Wang, *JHEP* **06** (2014), 128.
- [19] P. Baseilhac, *Nucl. Phys. B* **754** (2006), 309.
- [20] P. Baseilhac and K. Koizumi, *J. Stat. Mech.* (2007), P09006.

- [21] P. Baseilhac and S. Belliard, *Lett. Math. Phys.* **93** (2010), 213; *Nucl. Phys. B* **873** (2013), 550.
- [22] S. Belliard and N. Crampé, *SIGMA* **9** (2013), 072.
- [23] S. Belliard, *Nucl. Phys. B* **892** (2015), 1.
- [24] S. Belliard and R. A. Pimenta, *Nucl. Phys. B* **894** (2015), 527.
- [25] J. Avan, S. Belliard, N. Grosjean and R. A. Pimenta, *Nucl. Phys. B* **899** (2015), 229.
- [26] E. K. Sklyanin, *Lect. Notes Phys.* **226** (1985), 196; *J. Sov. Math.* **31** (1985), 3417; *Prog. Theor. Phys. Suppl.* **118** (1995), 35.
- [27] H. Frahm, A. Seel and T. Wirth, *Nucl. Phys. B* **802** (2008), 351.
- [28] G. Niccoli, *Nucl. Phys. B* **870** (2013), 397; *J. Phys. A* **46** (2013), 075003.
- [29] S. Faldella, N. Kitanine and G. Niccoli, *J. Stat. Mech.* (2014), P01011.
- [30] N. Kitanine, J.-M. Maillet and G. Niccoli, *J. Stat. Mech.* (2014), P05015.
- [31] P. Fazekas and P. W. Anderson, *Phil. Mag.* **30** (1974), 423.
- [32] For a review see: G. Misguich and C. Lhuillier, “Two-dimensional quantum anti-ferromagnets,” in *Frustrated Spin Systems*, ed. H. T. Diep, World Scientific, 2005, cond-mat/0310405.
- [33] C. Honerkamp and W. Hofstetter, *Phys. Rev. Lett.* **92** (2004), 170403.
- [34] W. Hofstetter, *Adv. Solid State. Phys.* **45** (2005), 109.
- [35] H. P. Büchler, M. Hermele, S. D. Huber, Matthew P. A. Fisher and P. Zoller, *Phys. Rev. Lett.* **95** (2005), 040402.
- [36] A. Onufriev and J. B. Marston, *Phys. Rev. B* **59** (1999), 12573.
- [37] K. Penc, M. Mambrini, P. Fazekas and F. Mila, *Phys. Rev. B* **68** (2003), 012408, and references therein.

- [38] V. Chari and A. Pressley, *A Guide to Quantum Groups*, Cambridge University Press, 1994.
- [39] I. V. Cherednik, *Theor. Math. Phys.* **43** (1980), 356.
- [40] D. V. Chudnovsky and G. V. Chudnovsky, *Phys. Lett. A* **79** (1980), 36.
- [41] C. L. Schultz, *Phys. Rev. Lett.* **46** (1981), 629.
- [42] O. Babelon, H. J. de Vega and C. M. Viallet, *Nucl. Phys. B* **190** (1981), 542.
- [43] J. H. H. Perk and C. L. Schultz, *Phys. Lett. A* **84** (1981), 407.
- [44] J. H. H. Perk and C. L. Schultz, “Families of commuting transfer matrices in q-state vertex models”, in *Non-linear integrable systems - classical theory and quantum theory*, eds. M. Jimbo and T. Miwa, World Scientific, 1983, pp. 135-152.
- [45] C. L. Schultz, *Physica A* **122** (1983), 71.
- [46] J. H. H. Perk and H. Au-Yang, “Yang-Baxter Equation”, in *Encyclopedia of Mathematical Physics*, eds. J. -P. Francoise, G. L. Naber and T. S. Tsun, Academic Press, 2006. Extended version in [arXiv:math-ph/0606053](https://arxiv.org/abs/math-ph/0606053).
- [47] V. V. Bazhanov, *Phys. Lett. B* **159** (1985), 321.
- [48] M. Jimbo, *Commun. Math. Phys.* **102** (1986), 537.
- [49] R. I. Nepomechie, *Lett. Math. Phys.* **62** (2002), 83.
- [50] H. J. de Vega, *Nucl. Phys. B* **240** (1984), 495.
- [51] M. T. Batchelor, R. J. Baxter, M. J. O’Rourke and C. M. Yung, *J. Phys. A* **28** (1995), 2759.
- [52] M. Karowski, *Nucl. Phys. B* **153** (1979), 244.
- [53] P. P. Kulish, N. Yu. Reshetikhin and E. K. Sklyanin, *Lett. Math. Phys.* **5** (1981), 393.
- [54] P. P. Kulish and E. K. Sklyanin, *Quantum spectral transform method: recent developments, Lecture Notes in Physics* **151** (1982), 61.

- [55] A.N. Kirillov and N.Yu. Reshetikhin, *J. Sov. Math.* **35** (1986), 2627; *J. Phys.* **A 20** (1987), 1565.
- [56] K. Hao, J. Cao, G.-L. Li, W.-L. Yang, K. Shi and Y. Wang, *JHEP* **05** (2016), 119.